Principles of Elasticity & Numerical Methods

A Foundational Guide for Engineering Students

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Course Outline

1. Fundamentals of Elasticity

Introduction: How Materials Behave

What is Elasticity?
Elasticity is the physical property of a material that allows it to return to its original shape and size after a deforming force is removed.

Real-World Examples:

- Stretching a rubber band.
- A diving board bending under a swimmer's weight and springing back.
- Car suspensions absorbing shock.

The Twin Pillars: Stress and Strain

To understand elasticity, we need to quantify forces and deformations.

Stress (σ)

- Internal resistance to an external force.
- Force per unit area.
- Formula: $\sigma = \frac{F}{A}$
- Units: Pascals (Pa) or N/m^2.

Strain (ϵ)

- The physical measure of deformation.
- Change in dimension over original dimension.
- Formula: $\epsilon = \frac{\Delta L}{L}$
- Units: Dimensionless (no units).

Strain-Displacement Relations (1D)

- Let's look inside the material. How do individual points move?
- We call this movement **displacement**.
- In a 1D bar aligned with the x-axis, let the displacement of any point be denoted by u.

The 1D Relation
Strain is the rate of change of displacement with respect to the length:

$$\epsilon_x = \frac{du}{dx}$$

- If u is constant (the whole bar just moves to the right), $\frac{du}{dx} = 0$. No stretching means no strain!

Strain-Displacement Relations (2D & 3D)

2D Case:

- Displacements in x and y directions are u and v.
- Normal strains: $\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_y = \frac{\partial v}{\partial y}$
- Shear strain (angle change): $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

3D Case:

- We add the z-axis and displacement w.
- The concept scales up! We get 3 normal strains $(\epsilon_x, \epsilon_y, \epsilon_z)$ and 3 shear strains $(\gamma_{xy}, \gamma_{yz}, \gamma_{zx})$.

Stress-Strain Relations: Hooke's Law

Hooke's Law (1D)
For materials in the elastic region, stress is directly proportional to strain.

$$\sigma = E \cdot \epsilon$$

- E is **Young's Modulus**.
- It represents the stiffness of the material.
- Steel has a very high E (hard to stretch). Rubber has a very low E (easy to stretch).

Stress-Strain Relations (3D Generalization)

- In 3D, pulling a material in the x-direction makes it thinner in the y and z directions. This is the **Poisson Effect**.
- Poisson's Ratio (ν): The ratio of lateral strain to axial strain.
- Because all directions affect each other, we use a 6×6 matrix (the Constitutive Matrix) to link the 6 stress components to the 6 strain components.

Student Note

You don't need to memorize the 6×6 matrix today! Just understand that in 3D, every force direction communicates with every deformation direction.

2. Plane Stress and Plane Strain

Simplifying Reality

- Solving full 3D elasticity problems by hand is incredibly difficult.
- Whenever possible, engineers make assumptions to reduce a 3D problem into a 2D problem.
- The two most common assumptions are **Plane Stress** and **Plane Strain**.

Plane Stress Condition

Definition
A state of stress where the normal stress and shear stresses directed perpendicular to the x-y plane are assumed to be zero.

- **When to use it:** Very **thin** objects loaded in their own plane.
- **Mathematical condition:** $\sigma_z = 0$, $\tau_{xz} = 0$, $\tau_{yz} = 0$.
- **Example:** A thin metal plate with a hole in it, being pulled from the sides. The surfaces are free to expand or contract, so no stress builds up in the thickness direction.

Plane Strain Condition

Definition
A state of strain where the normal strain and shear strains directed perpendicular to the x-y plane are assumed to be zero.

- **When to use it:** Very **long** or thick objects with a uniform cross-section.
- **Mathematical condition:** $\epsilon_z = 0$, $\gamma_{xz} = 0$, $\gamma_{yz} = 0$.
- **Example:** A long retaining wall, a water dam, or a long cylindrical pipe. The material is constrained by itself, so it cannot deform in the long z-direction.

3. Introduction to Numerical Methods

The Need for Numerical Methods

- **Analytical Solutions:** Give exact, continuous mathematical formulas.
- *The Catch:* They only work for simple geometries (perfect cylinders, simple rectangles) and simple loading conditions.
- **Real Engineering:** Mechanical parts have complex curves, holes, fillets, and experience multiple loads at once.
- **The Fix:** We use **Numerical Methods** to find highly accurate *approximate* solutions that computers can solve. (This is the math running under the hood of software like ANSYS!)

The Principle of Minimum Potential Energy

- Nature always seeks the path of least resistance.
- **Total Potential Energy (Π)** of a system is defined as:

$$\Pi = U - W$$

- Where:
  - U = Internal Strain Energy (energy stored like a spring).
  - W = Work done by external forces.

The Principle

Of all possible shapes a structure could take when deformed, the **true shape** is the one that makes the Total Potential Energy (Π) a minimum.

4. The Rayleigh-Ritz Method

What is the Rayleigh-Ritz Method?

- It is a direct method to find an approximate solution using the Minimum Potential Energy principle.
- Instead of solving complex differential equations, we **guess** the shape of the deformation.
- We write the guess as a polynomial with unknown coefficients.
- We then find the coefficients that minimize the energy.

Steps in Rayleigh-Ritz

1. **Assume a trial function** (a guess) for displacement: $u(x) = a_0 + a_1 x + a_2 x^2$
2. **Apply Boundary Conditions:** Adjust a_0, a_1, a_2 so the guess matches reality (e.g., $u = 0$ at a fixed wall).
3. **Formulate the Energy Equation:** Plug the trial function into $\Pi = U - W$.
4. **Minimize the Energy:** Take the derivative of Π with respect to the remaining unknown coefficients and set it to zero:

$$\frac{\partial \Pi}{\partial a_i} = 0$$

5. **Solve:** Find the values of a_i to get your final approximate equation!

5. The Galerkin Method

What is the Galerkin Method?

- Another powerful approximation technique, categorized as a **Weighted Residual Method**.
- **The core idea:** When you guess an approximate solution to a differential equation, it won't be perfect.
- The leftover error is called the **Residual (R)**.
- We can't make the residual zero everywhere. But we can make the *average* error zero across the whole domain!

Steps in the Galerkin Method

1. **Assume a trial function** (just like Rayleigh-Ritz).
2. **Calculate the Residual (R)** by plugging the guess into the governing differential equation.
3. **Apply Weighting Functions (W_i):** In Galerkin's method, the weighting functions are the same as the trial functions.
4. **Set the integral of the weighted residual to zero:**

$$\int_{Domain} W_i \cdot R \, dx = 0$$

5. **Solve** the resulting equations for your unknown coefficients.

Note: For solid mechanics problems, Rayleigh-Ritz and Galerkin often lead to the exact same system of equations!

6. Application to Simple Problems

1. Axially Loaded Members

- **The Setup:** A straight bar pulled or pushed along its length (x-axis).
- **Displacement:** 1D movement, $u(x)$.
- **Boundary Condition Example:** If fixed at the left wall ($x = 0$), then the displacement there is zero: $u(0) = 0$.
- **Strain Energy (U):**

$$U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$$

(where A is cross-sectional area, E is Young's modulus, L is length).

- **The Setup:** A beam fixed rigidly at one end and completely free at the other (like a balcony).
- **Deformation:** Bending (transverse displacement, $v(x)$).
- **Boundary Conditions at the wall ($x = 0$):**
  - Displacement is zero: $v(0) = 0$
  - Slope (rotation) is zero: $\frac{dv}{dx}(0) = 0$
- Any guessed trial function in Rayleigh-Ritz MUST satisfy these two conditions!

3. Simply Supported Beams

- **The Setup:** A beam resting on two supports (pin and roller) at either end (like a simple footbridge). Let length be L.
- **Boundary Conditions:**
  - At left support ($x = 0$): Displacement is zero, $v(0) = 0$.
  - At right support ($x = L$): Displacement is zero, $v(L) = 0$.
  - (Note: The beam is free to rotate at the ends, so slope is NOT zero here).
- A good trial function guess here is often a sine wave: $v(x) = A\sin(\frac{\pi x}{L})$, because sine is zero at both 0 and π.

Handling Different Types of Loads

How we calculate Work Done (W) depends on the load type:

Point Loads (P)

- A concentrated force.
- Work is simply Force $\times$ Displacement at that exact point.
- $W = P \cdot v(x_p)$

Distributed Loads (q)

- Force spread over a length (e.g., self-weight).
- Work is the integral of the load $\times$ displacement over the length.
- $W = \int_0^L q \cdot v(x)dx$

Summary & Conclusion

Summary of Key Takeaways

1. **Elasticity** describes material recovery via Stress (σ) and Strain (ϵ).
2. **Plane Stress & Strain** are powerful assumptions that reduce complex 3D math into solvable 2D math.
3. Exact solutions are rare in the real world. We rely on **Numerical Methods**.
4. **Rayleigh-Ritz** minimizes Total Potential Energy.
5. **Galerkin** minimizes the residual mathematical error.
6. Both methods require a good ”guess” (trial function) that respects the physical **Boundary Conditions** of the beam or member.

Questions?

Thank you for your attention!