



Mechanical Vibrations

Introduction to Two-Degree-of-Freedom Systems, Dynamic Vibration Absorber, Damping Materials, Vibration Modes and FEA Modal Analysis

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Core modules

- ▶ Basics of multi-degree-of-freedom vibration
- ▶ Governing equations of 2-DOF systems
- ▶ Natural frequencies and mode shapes
- ▶ Physical interpretation of normal modes
- ▶ Dynamic vibration absorber (DVA)

Advanced modules

- ▶ Damping materials and loss mechanisms
- ▶ Mode orthogonality and modal coordinates
- ▶ Finite element modal analysis workflow
- ▶ Model validation and engineering limitations
- ▶ Design insights for vibration control

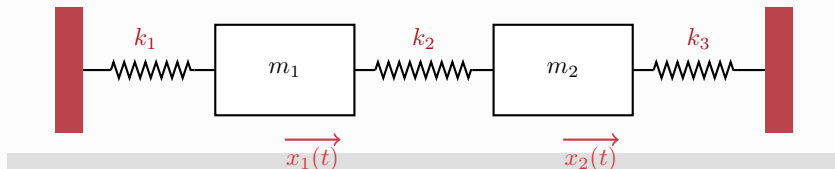
Why Two-Degree-of-Freedom Systems Matter

- ▶ A two-degree-of-freedom (2-DOF) system requires **two independent generalized coordinates** to completely describe its motion.
- ▶ Such systems are the simplest models that exhibit **coupled motion, multiple natural frequencies, and multiple vibration modes**.
- ▶ In free vibration, a 2-DOF system possesses **two natural frequencies** and **two normal modes**.
- ▶ Real engineering examples include engine–mount systems, machine foundations, vehicle quarter-car models, and coupled shaft–disc torsional systems.

Key idea

An SDOF model can predict resonance, but a 2-DOF model explains **mode interaction, phase relation, and how an attached absorber modifies system dynamics**.

Basic Mechanical Model of a 2-DOF System



- Coordinates $x_1(t)$ and $x_2(t)$ are measured from static equilibrium positions.
- The masses are dynamically coupled through the intermediate spring k_2 , which makes each equation depend on both coordinates.

Assumptions and Standard Notation

Assumptions

- ▶ Linear elastic springs
- ▶ Small displacement vibration
- ▶ Lumped parameter model
- ▶ Rigid masses
- ▶ Viscous damping if included

Symbols

- ▶ m_i : mass, in kg
- ▶ k_i : stiffness, in N/m
- ▶ c_i : damping coefficient, in N s/m
- ▶ x_i : displacement, in m
- ▶ ω_n : natural circular frequency, in rad/s
- ▶ ϕ : mode shape vector

Validity limit

The linear 2-DOF model is valid when geometric nonlinearity, frictional nonlinearity, and large-amplitude effects are negligible.

Equations of Motion by Newton's Second Law

For the undamped free vibration model,

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0$$

In matrix form,

$$\mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{0}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

- ▶ $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix.
- ▶ The off-diagonal terms of $\mathbf{K}$ indicate **static coupling** between the coordinates.

Harmonic Solution and Frequency Equation

Assume a harmonic solution

$$\mathbf{x}(t) = \mathbf{X} \sin(\omega t + \theta)$$

then

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{X} = \mathbf{0}$$

For a non-trivial solution,

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

Hence, for the 2-DOF system,

$$\begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 + k_3 - \omega^2 m_2 \end{vmatrix} = 0$$

Interpretation

The characteristic equation is quadratic in ω^2 , so a 2-DOF system yields **two natural frequencies** ω_1 and ω_2 .

Natural Frequencies and Mode Shapes

For each root ω_r , the modal vector ϕ_r is obtained from

$$(K - \omega_r^2 M) \phi_r = 0$$

Let

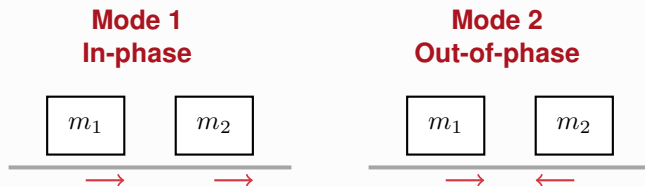
$$\phi_r = \begin{bmatrix} \phi_{1r} \\ \phi_{2r} \end{bmatrix}$$

Then only the **relative amplitude ratio** is fixed:

$$\frac{\phi_{2r}}{\phi_{1r}} = \frac{k_1 + k_2 - \omega_r^2 m_1}{k_2}$$

- ▶ The modal vector gives the deformation pattern, not the absolute amplitude.
- ▶ Any scalar multiple of a mode shape represents the same physical mode.
- ▶ In practice, mode shapes are normalized for numerical efficiency.

Physical Meaning of the Two Modes



- ▶ In the lower mode, both masses usually move in the **same direction**.
- ▶ In the higher mode, the masses move in **opposite directions**, causing larger deformation in the coupling spring.

General Free-Vibration Response

The complete free response is the superposition of two modal contributions:

$$\mathbf{x}(t) = A_1 \boldsymbol{\phi}_1 \sin(\omega_1 t + \theta_1) + A_2 \boldsymbol{\phi}_2 \sin(\omega_2 t + \theta_2)$$

- ▶ The constants $A_1, A_2, \theta_1, \theta_2$ depend on initial conditions.
- ▶ Arbitrary initial excitation generally excites **both modes simultaneously**.
- ▶ Measured vibration may exhibit beating and coordinate-dependent amplitude variation.

Engineering insight

Multi-mode superposition explains why even simple coupled machines can show complex vibration signatures.

Orthogonality of Modes

For symmetric $\mathbf{M}$ and $\mathbf{K}$, the normal modes are orthogonal:

$$\phi_r^T \mathbf{M} \phi_s = 0, \quad \phi_r^T \mathbf{K} \phi_s = 0, \quad r \neq s$$

- ▶ Orthogonality is fundamental for transforming coupled equations into uncoupled modal equations.
- ▶ Mass-normalization ($\phi_r^T \mathbf{M} \phi_r = 1$) is used extensively in FEA software.
- ▶ This forms the basis of modal expansion for complex structures.

From 2-DOF to Dynamic Vibration Absorber

- ▶ A dynamic vibration absorber converts an original SDOF primary system into an effective **2-DOF coupled system**.
- ▶ An auxiliary mass–spring subsystem is attached to the main vibrating system.
- ▶ Proper tuning redistributes the resonance peak into two nearby resonances.

Dynamic Vibration Absorber System Schematic

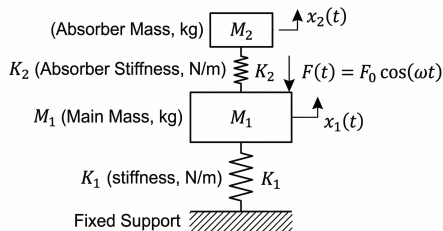


Figure: Schematic of a Primary System with an attached Auxiliary Absorber

Principle of the Dynamic Vibration Absorber

For a harmonically excited primary system, an absorber is tuned such that

$$\omega_a = \sqrt{\frac{k_a}{m_a}} \approx \omega_f$$

where

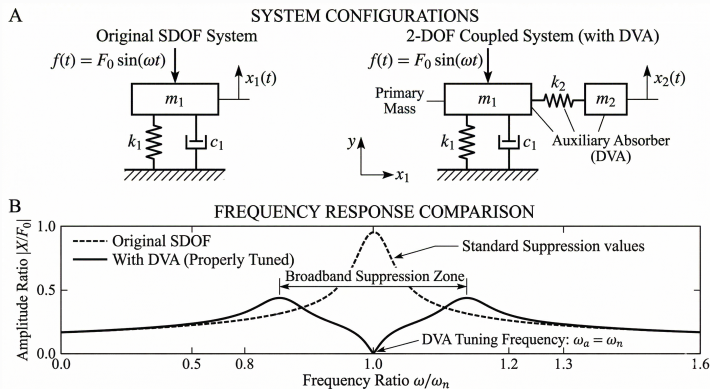
- ▶ m_a : absorber mass (kg), k_a : absorber stiffness (N/m)
- ▶ ω_a : absorber natural circular frequency (rad/s)
- ▶ ω_f : forcing circular frequency (rad/s)

Operating principle

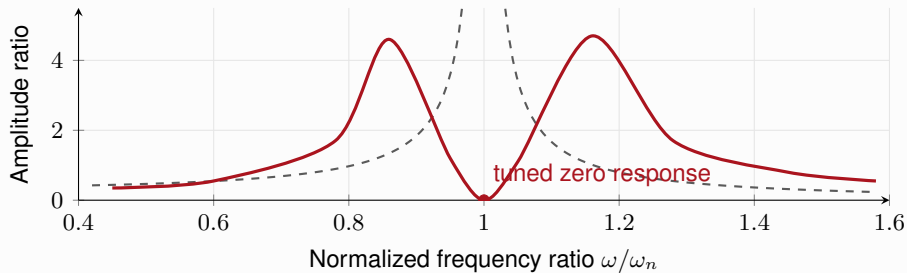
When the absorber is tuned to the excitation frequency, the primary mass response can theoretically become **zero** in the ideal undamped case.

How the DVA Changes the Frequency Response

- ▶ Without the absorber, the main system has one dominant resonance.
- ▶ After attachment, the system becomes 2-DOF and exhibits **two resonance frequencies**.
- ▶ The original resonance frequency becomes an anti-resonance point for the primary mass.
- ▶ In the ideal undamped case, the main mass displacement is zero at the tuned frequency.



Classical DVA Response Sketch



Dashed curve: primary system without absorber; red curve: response with tuned absorber.

Advantages and Limitations of DVA

Advantages

- ▶ Effective for narrow-band harmonic excitation
- ▶ Passive and relatively simple
- ▶ Useful in buildings and rotating systems

Limitations

- ▶ Sensitive to mistuning
- ▶ Undamped absorber introduces high side peaks
- ▶ Startup can cross new resonances

Practical note

A finite amount of damping is often introduced to control side resonances, even if it slightly reduces the "zero response" efficiency.

Damped Absorber and Practical Design Insight

In practice, the coupled equations become:

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

- ▶ Damping broadens the suppression bandwidth and limits excessive absorber motion.
- ▶ Design involves a trade-off between **peak suppression** and **bandwidth**.

Design parameters

Mass ratio $\mu = m_a/m$, tuning ratio $f = \omega_a/\omega_n$, and absorber damping ratio ζ_a .

Vibration Damping Materials: Need and Function

Energy Dissipation Mechanism

- ▶ **Viscoelasticity:** Combines elastic (storage) and viscous (dissipative) properties.
- ▶ **Hysteresis:** Under cyclic loading, the stress-strain curve forms a loop; the area inside represents **energy lost as heat** per cycle.
- ▶ **Phase Lag (δ):** In these materials, strain lags behind stress by a phase angle, leading to the complex modulus $E^* = E' + iE''$.

Hysteresis Loop for a Viscoelastic Material: Stress-Strain Relationship

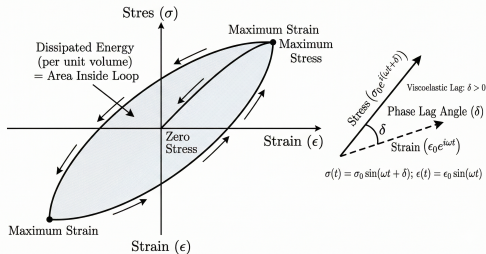


Figure: Hysteresis loop in Viscoelastic Materials

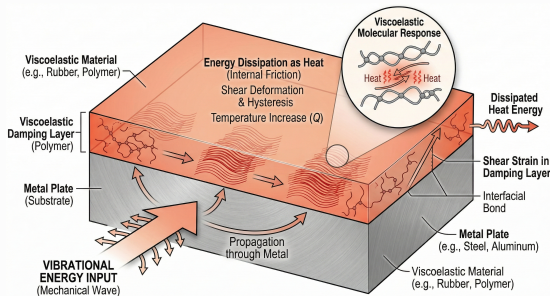
Classification of Damping Materials

Material classes

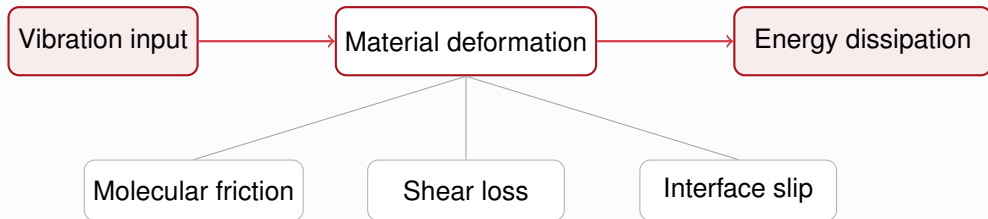
- ▶ Viscoelastic polymers
- ▶ Elastomers and rubber
- ▶ Constrained-layer laminates
- ▶ Fiber composites

Application forms

- ▶ Free-layer damping
- ▶ Constrained-layer damping
- ▶ Damping coatings
- ▶ Mounts and bushings



Damping Mechanisms in Materials



High damping is associated with phase lag between stress and strain.

Viscoelastic Material Model

For a linear viscoelastic material under harmonic loading:

$$E^* = E' + iE''$$

where E' is the storage modulus and E'' is the loss modulus.

The material loss factor is:

$$\eta = \frac{E''}{E'}$$

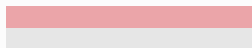
Meaning

A larger η indicates greater energy dissipation per cycle.

Free-Layer and Constrained-Layer Damping

Free-layer damping

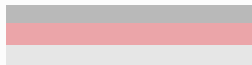
- ▶ Layer bonded to base
- ▶ Effective in bending
- ▶ Simpler construction



Free-layer

Constrained-layer damping

- ▶ Core sandwiched by stiff layer
- ▶ Energy dissipation in shear
- ▶ Higher effectiveness



Constrained-layer

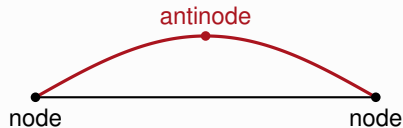
What are Vibration Modes?

- ▶ A vibration mode is a characteristic deformation pattern at one natural frequency.
- ▶ Associated with an eigenvalue–eigenvector pair.
- ▶ Lower modes usually dominate the dynamic response.
- ▶ Modal analysis extracts these shapes to avoid resonance in design.

Mode Shapes, Nodes and Participation

Important terms

- ▶ **Antinode**: Point of max displacement
- ▶ **Node**: Point of zero displacement
- ▶ **Modal participation**: Contribution to response



Modal Coordinates and Decoupling

Express physical displacement as $\mathbf{x}(t) = \Phi \mathbf{q}(t)$.

Substituting into $M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$ and using orthogonality:

Benefit

This transformation converts coupled physical coordinates into independent single-mode coordinates, simplifying complex system analysis.

The Governing Eigenproblem

FEA reduces a continuous structure to a finite number of DOFs via discretization:

$$(K - \omega_i^2 M) \phi_i = 0$$

- ▶ **Eigenvalues** (ω_i^2): The roots represent the natural frequencies.
- ▶ **Eigenvectors** (ϕ_i): Represent the **Normal Modes** or displacement patterns.
- ▶ **Mass Participation**: Not all modes are equally important; we look for modes with high *Effective Modal Mass*.
- ▶ **Normalization**: Modes are usually **Mass-Normalized** such that $\phi_i^T M \phi_i = 1$.

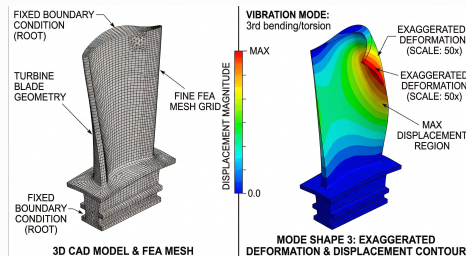
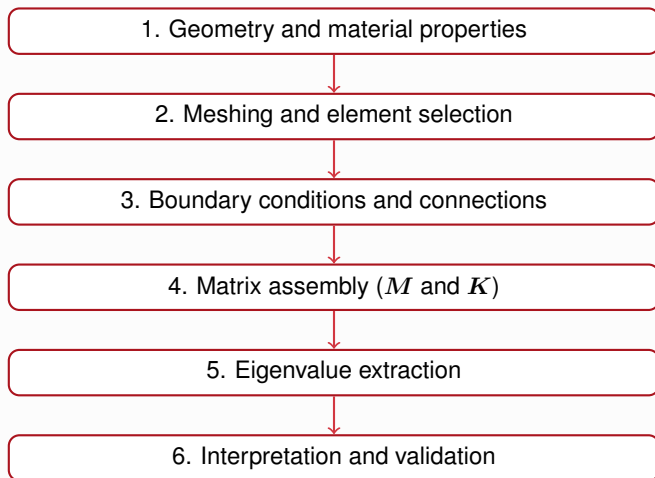


Figure: Higher-order mode shape of a turbine blade.

FEA Modal Analysis Workflow



What FEA Modal Analysis Can and Cannot Do

Can do

- ▶ Predict resonance frequencies
- ▶ Identify critical deformation
- ▶ Check connectivity

Cannot do directly

- ▶ Give true forcing amplitude
- ▶ Predict fatigue life
- ▶ Guarantee correlation without test

Important

Modal analysis alone gives mode shapes; displacement magnitude under real excitation requires harmonic or transient analysis.

Theory

- ▶ 2-DOF explains coupling
- ▶ DVA modifies eigenstructure
- ▶ Damping controls peaks

Practice

- ▶ FEA extracts modes
- ▶ Testing validates simulation
- ▶ Design combines tuning and damping

Design philosophy

Effective vibration control combines **dynamic modeling**, **material damping**, and **validated simulation**.

References

- ① Two degree of freedom systems lecture notes, Istanbul Technical University.
- ② Dynamic vibration absorber theory, Penn State Acoustics Lab.
- ③ FEA modal analysis overview, CAEflow.
- ④ 2-DOF free-vibration teaching material, University of Alberta.

Thank You

Noise Vibration and harshness – Lecture 12 Module 3

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