Forced Vibration with Harmonic Excitation

Resonance, Damped and Undamped System Response

Noise, Vibration and Harshness

Course Code: BME4208

Dinesh Kumar
Assistant Professor, SoCE
Dr. D. Y. Patil Institute of University, Pune

February 24, 2026

Outline

1. Introduction to Forced Vibration
2. Response of Undamped Systems
3. Response of Damped Systems
4. Practical Applications and Design
5. Summary and Key Takeaways

What is Forced Vibration?

Definition:

- Vibration caused by *external time-varying forces*
- System responds to applied excitation
- Energy continuously supplied from external source
- Response depends on excitation characteristics

Types of Excitation:

- **Harmonic** - Sinusoidal forces
- **Periodic** - Non-sinusoidal repeating
- **Transient** - Short duration impulse
- **Random** - Non-deterministic

Key Characteristics

- ✓ Steady-state response
- ✓ Transient response
- ✓ Frequency-dependent
- ✓ Amplitude modulation

Focus of This Lecture

Harmonic excitation - the most fundamental and important type in NVH analysis

Harmonic Excitation

Mathematical Form:

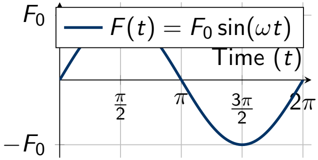
$$F(t) = F_0 \sin(\omega t) \quad \text{or} \quad F(t) = F_0 \cos(\omega t)$$

Where:

- F_0 = Force amplitude (N)
- ω = Excitation frequency (rad/s)
- t = Time (s)

Why Study Harmonic Excitation?

- Forms basis for understanding complex vibrations
- Many real sources are harmonic
- Fourier analysis reduces complex to harmonic



Common Sources

- Rotating machinery unbalance
- Engine firing frequencies
- Electric motor harmonics
- Road surface undulations

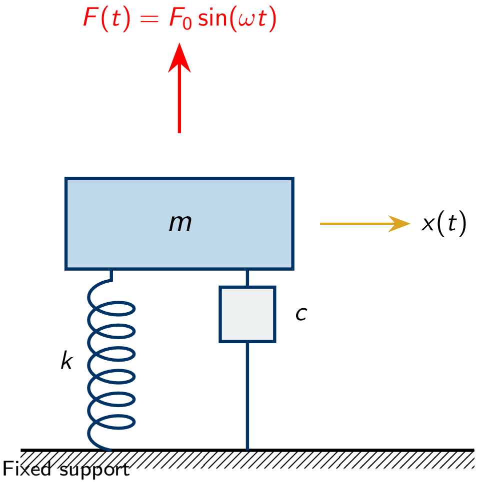
Free vs Forced Vibration

Aspect	Free Vibration	Forced Vibration
Energy Source	Initial disturbance only	Continuous external force
Frequency	Natural frequency ω_n	Excitation frequency ω
Response	Decays with damping	Sustained steady-state
Amplitude	Depends on initial conditions	Depends on force magnitude
Equation	$m\ddot{x} + c\dot{x} + kx = 0$	$m\ddot{x} + c\dot{x} + kx = F(t)$
Duration	Temporary (damped out)	Continuous (as long as force applied)
Application	Natural frequency measurement	Operating condition analysis
Critical Issue	Damping ratio	Resonance avoidance

Key Insight

In **forced vibration**, the system vibrates at the *excitation frequency* ω, NOT at the natural frequency ω_n (except during initial transient).

System Model for Analysis



Equation of Motion

Key Parameters

Important Frequency Parameters

Natural Frequency:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{(rad/s)}$$

$$f_n = \frac{\omega_n}{2\pi} \quad \text{(Hz)}$$

Damping Ratio:

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$$

Frequency Ratio:

$$r = \frac{\omega}{\omega_n}$$

The frequency ratio r is the single most important parameter in forced vibration

Damping Classifications

- $\zeta = 0$: Undamped
- $0 < \zeta < 1$: Underdamped
- $\zeta = 1$: Critically damped
- $\zeta > 1$: Overdamped

Frequency Ratio Regions

- $r < 1$: Below resonance
- $r = 1$: At resonance
- $r > 1$: Above resonance

Undamped System - Equation of Motion

Governing Equation ($c = 0$):

$$m\ddot{x} + kx = F_0 \sin(\omega t)$$

Dividing by m:

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin(\omega t)$$

General Solution:

$$x(t) = x_h(t) + x_p(t)$$

Where:

- $x_h(t)$ = Homogeneous (transient) solution
- $x_p(t)$ = Particular (steady-state) solution

Homogeneous Solution

$$x_h(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

- Depends on initial conditions
- Vibrates at natural frequency
- Dies out in damped systems

Particular Solution

$$x_p(t) = X \sin(\omega t)$$

- Steady-state response

Steady-State Response - Undamped System

Assumed solution form:

$$x_p(t) = X\sin(\omega t)$$

Taking derivatives:

$$\dot{x}_p = X\omega\cos(\omega t)$$
$$\ddot{x}_p = -X\omega^2\sin(\omega t)$$

Substituting into equation of motion:

$$-X\omega^2\sin(\omega t) + \omega_n^2 X\sin(\omega t) = \frac{F_0}{m}\sin(\omega t)$$

$$X(\omega_n^2 - \omega^2) = \frac{F_0}{m}$$

Amplitude of Response

$$X = \frac{F_0/m}{\omega_n^2 - \omega^2} = \frac{F_0/k}{1 - (\omega/\omega_n)^2} = \frac{\delta_{st}}{1 - r^2}$$

Magnification Factor - Undamped System

Magnification Factor (Dynamic Amplification):

$$M = \frac{X}{\delta_{st}} = \frac{1}{1 - r^2}$$

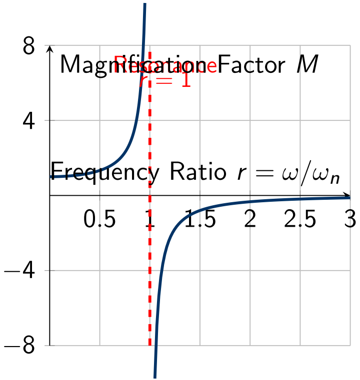
Behavior by Frequency Range

Low frequency ($r \ll 1$):

- $M \approx 1$
- $X \approx \delta_{st}$
- Quasi-static response

Near resonance ($r \approx 1$):

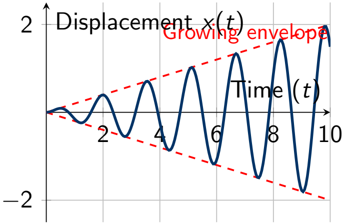
- $M \to \infty$
- Amplitude becomes very large
- System failure possible



Critical Observation

What is Resonance?

- Occurs when $\omega = \omega_n$ (i.e., $r = 1$)
- Excitation frequency matches natural frequency
- Amplitude grows without bound in undamped systems
- Most dangerous condition for structures



Mathematical Insight:

At resonance, the standard solution fails because:

$$X = \frac{\delta_{st}}{1 - r^2} \to \infty \quad \text{as } r \to 1$$

Engineering Implication

Resonance MUST be avoided in design by:

- Changing system stiffness/mass
- Adding damping
- Operating away from ω_n

Complete Solution - Undamped System

Total Response:

$$x(t) = x_h(t) + x_p(t) = A\cos(\omega_n t) + B\sin(\omega_n t) + X\sin(\omega t)$$

With initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$x(t) = x_0\cos(\omega_n t) + \frac{v_0}{\omega_n}\sin(\omega_n t) + \frac{\delta_{st}}{1-r^2}[\sin(\omega t) - r\sin(\omega_n t)]$$

Response Characteristics

Transient part: $x_h(t)$

- Oscillates at ω_n
- Depends on initial conditions
- Persists forever (no damping)

Steady-state part: $x_p(t)$

- Oscillates at ω

Beat Phenomenon

When $\omega \approx \omega_n$ (but $\omega \neq \omega_n$):

- Two frequencies close together
- Creates beating pattern
- Amplitude modulation occurs
- Period of beats: $T_b = \frac{2\pi}{|\omega_n - \omega|}$

Damped System - Equation of Motion

Governing Equation:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$$

Dividing by m and using $\omega_n^2 = k/m$ and $2\zeta\omega_n = c/m$:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin(\omega t)$$

General Solution:

$$x(t) = x_h(t) + x_p(t)$$

Key difference from undamped:

- $x_h(t)$ decays exponentially
- Only $x_p(t)$ survives long-term

Homogeneous Solution

For underdamped ($\zeta < 1$):

$$x_h(t) = e^{-\zeta\omega_n t}[A\cos(\omega_d t) + B\sin(\omega_d t)]$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

- Decays with time constant $\tau = \frac{1}{\zeta\omega_n}$
- Oscillates at damped frequency ω_d
- Becomes negligible after $3-4\tau$

Engineering Focus

For steady-state analysis, we focus on **particular solution** $x_p(t)$ only.

Steady-State Solution - Damped System

Assumed solution form (with phase lag):

$$x_p(t) = X \sin(\omega t - \phi)$$

Where:

- X = Amplitude of steady-state response
- ϕ = Phase angle (lag) between force and response

After substitution and algebraic manipulation:

Amplitude of Response

$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Phase Angle

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

Magnification Factor (Dynamic Amplification):

$$M = \frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Key Features

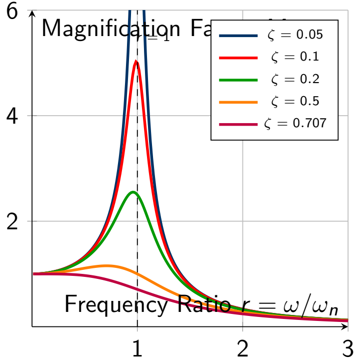
At low frequency ($r \ll 1$):

- $M \approx 1$ (all ζ)
- Static response

At $r = 1$ (resonance):

$$M = \frac{1}{2\zeta}$$

- Finite peak (unlike undamped)
- Damping limits amplitude



Peak Response and Resonant Frequency

Finding the peak response:

Taking $\frac{dM}{dr} = 0$ gives peak at:

$$r_{peak} = \sqrt{1 - 2\zeta^2}$$

Maximum magnification factor:

$$M_{max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Important Observations

- Peak occurs **before** $r = 1$ (for $\zeta < 0.707$)
- At $r = 1$: $M = \frac{1}{2\zeta}$ (not the maximum!)
- For $\zeta > 0.707$: No peak, monotonic decrease
- As $\zeta \to 0$: $r_{peak} \to 1$ and $M_{max} \to \infty$

ζ	r_{peak}	M_{max}
0.05	0.995	10.01
0.10	0.990	5.03
0.20	0.980	2.55
0.30	0.954	1.75
0.50	0.866	1.15
0.707	0.0	1.00

Critical Value

Phase Angle Analysis

Phase relationship between force and response:

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

Phase Behavior

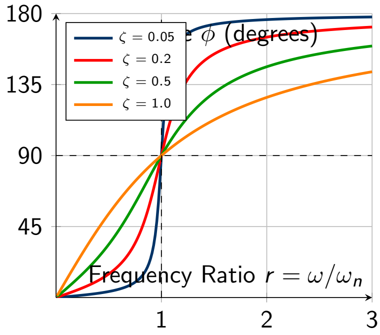
Low frequency ($r \ll 1$):

- $\phi \approx 0$
- Force and response in phase
- Response follows force

At resonance ($r = 1$):

- $\phi = 90$ (all ζ)
- Response lags force by quarter cycle
- Velocity in phase with force

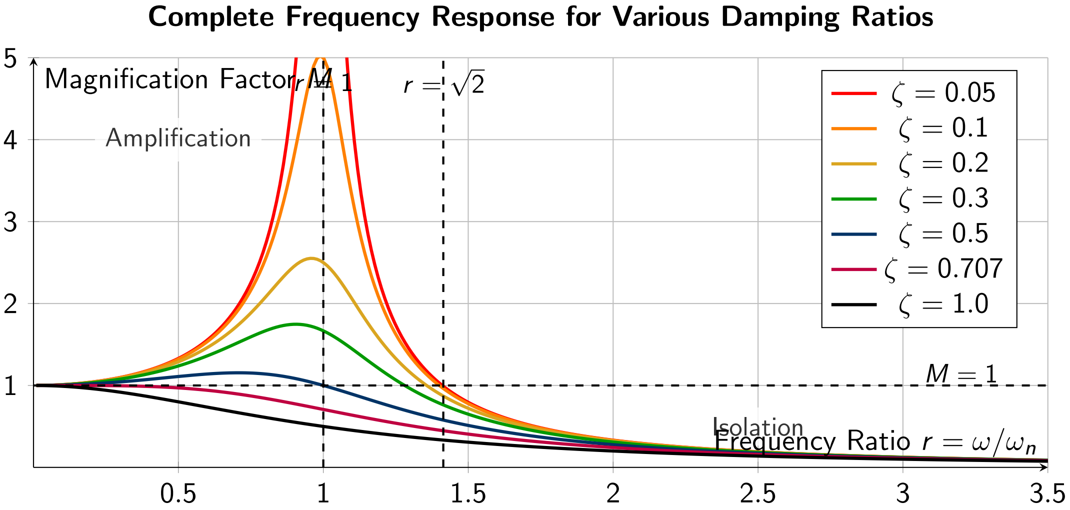
High frequency ($r \gg 1$):



Key Insight

Phase angle **always** passes through 90° at

Frequency Response Curves



Design Regions

Quality Factor and Bandwidth

Quality Factor (Q-factor):

$$Q = \frac{1}{2\zeta} = M_{at\ r=1}$$

Represents:

- Sharpness of resonance peak
- Energy stored vs. energy dissipated per cycle
- Selectivity of system

Half-Power Bandwidth:

Frequencies where $M = \frac{M_{max}}{\sqrt{2}}$:

$$r_1 \approx 1 - \zeta, \quad r_2 \approx 1 + \zeta \quad \text{(for small } \zeta\text{)}$$

$$\text{Bandwidth: } \Delta r = r_2 - r_1 \approx 2\zeta$$

Implications

Applications

Transmissibility and Force Transmission

Force transmitted to foundation:

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + (2\zeta r)^2}$$

Transmissibility Ratio:

$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

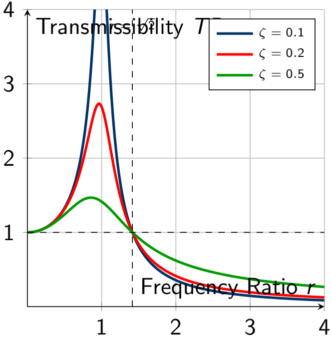
Key Characteristics

At $r = \sqrt{2}$:

- $TR = 1$ for all damping ratios
- Force transmitted = applied force
- Critical transition point

For $r > \sqrt{2}$:

- $TR < 1$ (isolation achieved)
- Lower damping gives better isolation



Rotating Unbalance Excitation

Common source of harmonic excitation in machinery

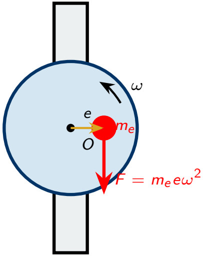
Unbalance Force

$$F(t) = m_e e\omega^2 \sin(\omega t)$$

where:

- m_e = eccentric mass
- e = eccentricity
- ω = rotation speed

Note: Force amplitude $F_0 = m_e e\omega^2$ increases with speed squared!



Dimensionless Plot

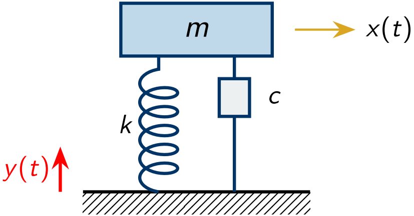
Plot $\frac{mX}{m_e e}$ vs r:

- At low speed: $X \approx 0$
- Near resonance: large amplitude
- At high speed: $X \approx e$ (constant)

Steady-state amplitude:

$$X = \frac{m_e e r^2}{\;} \cdot \frac{1}{\sqrt{\;}}$$

System subjected to motion at the base



Base Motion

$$y(t) = Y\sin(\omega t)$$

Equation of motion:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

Relative displacement $z = x - y$:

$$\frac{Z}{Y} = \frac{r^2\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Absolute displacement:

$$\frac{X}{Y} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Applications

Vibration Isolation Principles

Isolation Efficiency:

$$\text{Isolation} = (1 - TR) \times 100\%$$

Design Strategies:

1. Operate at high frequency ratio

- $r > 2$ to 3 minimum
- Lower natural frequency
- Use soft mounts/isolators

2. Control damping

- Low damping for steady-state isolation
- Adequate damping for startup/shutdown
- Trade-off required

r	$\zeta = 0.1$	$\zeta = 0.2$
1.5	1.73	1.46
2.0	0.57	0.62
2.5	0.30	0.34
3.0	0.19	0.21
4.0	0.11	0.12
5.0	0.07	0.08

Transmissibility values

Typical Isolator Systems

- **Machinery mounts:** $r = 3$ to 5
- **Vehicle suspension:** $r = 2$ to 3
- **Building isolation:** $r = 5$ to 10
- **Precision equipment:** $r = 10$ to 20

Example 1: Electric Motor on Flexible Foundation

Given:

- Motor mass: $m = 100$ kg
- Operating speed: $N = 1500$ rpm $= 25$ Hz
- Foundation stiffness: $k = 5 \times 10^5$ N/m
- Damping ratio: $\zeta = 0.15$

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5 \times 10^5}{100}} = 70.7 \text{ rad/s} = 11.25 \text{ Hz}$$

$$r = \frac{f}{f_n} = \frac{25}{11.25} = 2.22$$

$$M = \frac{1}{\sqrt{(1 - 2.22^2)^2 + (2 \times 0.15 \times 2.22)^2}} = 0.49$$

Conclusion

Design Examples - Unbalance Response

Example 2: Rotating Machine with Unbalance

Given:

- Total mass: $m = 50$ kg, Unbalance mass: $m_e = 0.5$ kg
- Eccentricity: $e = 10$ mm, Damping ratio: $\zeta = 0.1$
- Natural frequency: $f_n = 10$ Hz
- Operating speed: $N = 600$ rpm $= 10$ Hz

At resonance ($r = 1$):

$$X = \frac{m_e e}{m} \cdot \frac{r^2}{2\zeta} = \frac{0.5 \times 0.01}{50} \cdot \frac{1}{2 \times 0.1} = 0.5 \text{ mm}$$

Wait! This is at $r = 1$. Let's recalculate...

$$X = \frac{m_e e}{m} \cdot \frac{1}{2\zeta} = \frac{0.5 \times 0.01}{50} \cdot \frac{1}{0.2} = 0.25 \text{ mm}$$

Design Recommendation

Operating at resonance! Change natural frequency to $f_n = 5$ Hz (soft mounting) to achieve

Real-World Applications

Automotive Engineering

Engine mounting system:

- Engine idle: 600-900 rpm
- Firing frequency: 20-30 Hz (4-cyl)
- Mount natural frequency: 8-12 Hz
- Target: $r = 2$ to 3

Vehicle suspension:

- Body natural frequency: 1-1.5 Hz
- Wheel hop frequency: 10-15 Hz
- Road excitation: 0.5-20 Hz

Industrial Machinery

HVAC Systems

Air handling units:

- Fan speed: 600-1200 rpm
- Blade passing frequency consideration
- Spring isolators: $f_n = 3$ to 5 Hz
- Typical efficiency: 90-95%

Precision Equipment

Optical tables & microscopes:

- Building vibration: 1-50 Hz
- Isolation: $f_n < 1$ Hz
- Active control often used
- Target: 99% isolation

Key Equations Summary

Parameter	Equation
Equation of Motion	$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$
Natural Frequency	$\omega_n = \sqrt{k/m}, \quad f_n = \omega_n/(2\pi)$
Damping Ratio	$\zeta = c/(2\sqrt{km}) = c/(2m\omega_n)$
Frequency Ratio	$r = \omega/\omega_n$
Static Deflection	$\delta_{st} = F_0/k$
Magnification Factor	$M = 1/\sqrt{(1-r^2)^2 + (2\zeta r)^2}$
Phase Angle	$\phi = \tan^{-1}(2\zeta r/(1-r^2))$
Peak Response	$M_{max} = 1/(2\zeta\sqrt{1-\zeta^2})$ at $r_{peak} = \sqrt{1-2\zeta^2}$
At Resonance ($r = 1$)	$M = 1/(2\zeta), \quad \phi = 90$
Transmissibility	$TR = \sqrt{1+(2\zeta r)^2}/\sqrt{(1-r^2)^2 + (2\zeta r)^2}$
Quality Factor	$Q = 1/(2\zeta)$
Bandwidth	$\Delta r \approx 2\zeta$ (for small ζ)

Design Guidelines

Resonance Avoidance

1. Identify all excitation frequencies
2. Calculate system natural frequencies
3. Ensure $r \neq 1$ during operation
4. If unavoidable, pass through quickly
5. Add damping for transient suppression

Amplitude Reduction

If forced to operate near resonance:

- **Increase damping** significantly
- Add tuned mass dampers
- Use active control systems
- Structural modifications

Vibration Isolation

Target: $r > 2$ **to 3**

- Use soft mounts (low k)
- Increase mass if feasible
- Minimize damping for isolation
- But adequate ζ for startup

Critical Frequency Ratios

- $r = 1$: Resonance (avoid!)
- $r = \sqrt{2}$: $TR = 1$ transition
- $r < \sqrt{2}$: Amplification region
- $r > \sqrt{2}$: Isolation region
- $r > 3$: Good isolation

Comparison: Undamped vs Damped Systems

Characteristic	Undamped ($\zeta = 0$)	Damped ($\zeta > 0$)
Amplitude	$X = \delta_{st}/(1 - r^2)$	$X = \delta_{st}/\sqrt{(1 - r^2)^2 + (2\zeta r)^2}$
At Resonance	$X \to \infty$	$X = \delta_{st}/(2\zeta)$ (finite)
Phase Angle	0° ($r < 1$), 180° ($r > 1$)	Continuous: $0^\circ \to 90^\circ \to 180^\circ$
Transient Response	Persists forever	Decays exponentially
Peak Location	Exactly at $r = 1$	At $r = \sqrt{1 - 2\zeta^2}$ ($\zeta < 0.707$)
Beat Phenomenon	Yes (near $r = 1$)	Suppressed by damping
Isolation	Poor	Good (with proper design)
Practical Reality	Idealization only	All real systems have damping

Key Insight

Damping is crucial for:

- Limiting resonance amplitude
- Dissipating transient energy
- Providing stable steady-state response

Practical Measurement Techniques

Experimental Modal Analysis:

1. Impact Testing

- Use instrumented hammer
- Measure force and response
- Extract frequency response function (FRF)
- Identify natural frequencies and damping

2. Shaker Testing

- Controlled harmonic excitation
- Sweep through frequency range
- Measure amplitude and phase
- Plot frequency response curves

3. Operational Modal Analysis

- Ambient vibration measurement

Key Parameters from Testing:

From FRF Peak

- Natural frequency: f_n
- Damping ratio: from peak sharpness
- Mode shape: multiple measurement points

Damping Estimation

Half-power bandwidth method:

$$\zeta = \frac{f_2 - f_1}{2f_n}$$

where f_1, f_2 are frequencies at $M_{max}/\sqrt{2}$

Common Design Mistakes to Avoid

Mistake 1: Ignoring Resonance

Problem:

- Operating at or near $r = 1$
- Assuming damping will help enough

Solution:

- Always check frequency ratios
- Design for $r < 0.5$ or $r > 2$
- Campbell diagrams for varying speeds

Mistake 2: Excessive Damping

Problem:

- High damping reduces isolation
- Force transmission increases

Mistake 3: Insufficient Frequency Separation

Problem:

- r too close to 1 ($r = 1.2$ to 1.8)
- Still in amplification region

Solution:

- Target $r < 0.7$ or $r > 2.5$
- Account for tolerances and aging
- Consider temperature effects

Mistake 4: Neglecting Higher Modes

Problem:

- Focus only on first mode

Advanced Topics (Beyond This Lecture)

Multi-DOF Systems

- Multiple natural frequencies
- Mode shapes and modal analysis
- Modal superposition method
- Coupled vibrations

Control Strategies

- Passive control (isolators, dampers)
- Active control (actuators, feedback)
- Semi-active control (adaptive)
- Tuned mass dampers
- Dynamic vibration absorbers

Nonlinear Systems

- Hardening/softening springs
- Nonlinear damping
- Jump phenomena
- Subharmonic and superharmonic response

Computational Methods

- Finite element analysis (FEA)
- Transfer matrix method
- Numerical integration
- Frequency domain analysis

Conclusion

What We Learned:

- **Forced vibration** occurs when external forces continuously supply energy to a system
- System vibrates at *excitation frequency* ω, not natural frequency ω_n
- **Frequency ratio** $r = \omega/\omega_n$ is the most critical parameter
- **Resonance** $(r = 1)$ causes infinite amplitude in undamped systems, but damping limits it
- **Damping** provides both benefits (limits resonance) and drawbacks (reduces isolation)
- For **isolation**, operate at $r > 2$ with low damping
- **Transmissibility** analysis crucial for foundation force transmission

Design Philosophy

1. Identify excitation frequencies
2. Calculate natural frequencies
3. Ensure adequate separation
4. Balance damping requirements
5. Verify through analysis/testing
6. Monitor in operation

Remember

Prevention (proper design) is always better than **cure** (adding damping/control later).

References and Further Reading

1. Rao, S. S., *Mechanical Vibrations*, 6th Edition, Pearson, 2017
2. Inman, D. J., *Engineering Vibration*, 4th Edition, Pearson, 2013
3. Thomson, W. T. and Dahleh, M. D., *Theory of Vibration with Applications*, 5th Edition, Prentice Hall, 1998
4. Meirovitch, L., *Fundamentals of Vibrations*, McGraw-Hill, 2001
5. Den Hartog, J. P., *Mechanical Vibrations*, Dover Publications, 1985 (Classic text)
6. Harris, C. M. and Piersol, A. G., *Harris' Shock and Vibration Handbook*, 6th Edition, McGraw-Hill, 2009

Online Resources:

- MIT OpenCourseWare: Dynamics and Vibration
- NPTEL Video Lectures on Mechanical Vibrations
- Vibrationdata.com (Technical articles and software)

Standards:

Thank You!

Questions?

Dinesh Kumar
Assistant Professor, School of Civil Engineering
Dr. D. Y. Patil Institute of University, Pune

Course: BME4208 - Noise, Vibration and Harshness
B.Tech. Mechanical Engineering - Smart Manufacturing